Polynomial Dynamics

"A polynomial is a holo map from the Riemann sphere to itself which has a superattracting fixed point of highest possible ramification index at ∞."

$$K(f) := \{ z : (f^n(z))_{n \geq 0} \text{ is bounded} \}$$

$\hat{\mathbb{C}} \setminus K(f)$ is the Fatou component containing infinity.

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$$

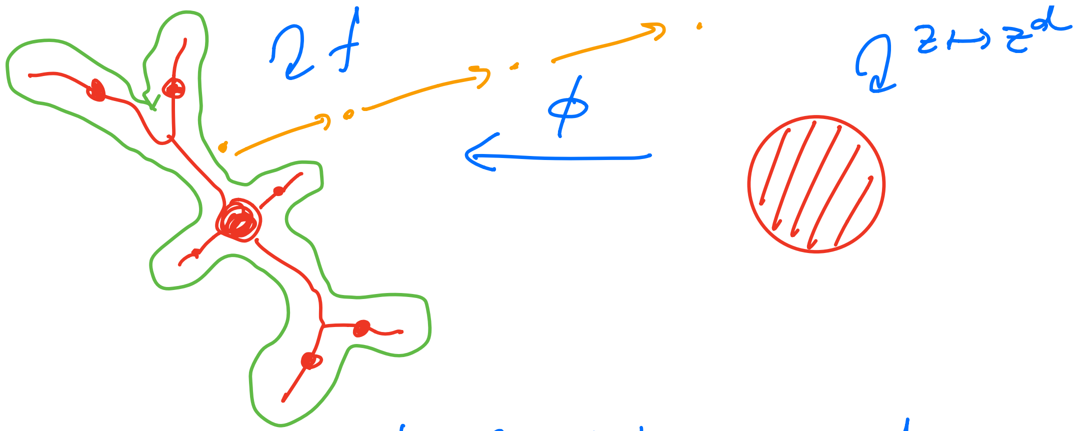
There are $(d-1)$ critical points counted with multiplicity.

Green function (potential function)

$$G(z) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

$$G(z) = 0 \longleftrightarrow z \in K(f)$$

$$\Delta G(z) = 0 \quad \text{if } z \notin K(f)$$



Böttcher: $\quad \phi \circ f \circ \phi^{-1}(w) = w^d \quad (|w| \to \infty)$

$$G(z) = \log|\phi(w)| \approx \log|w|$$

$$\log \phi = \underbrace{\log|\phi|}_{G} + i \underbrace{\arg \phi}_{\text{external angles}}$$

$$\phi \circ f^n \circ \phi^{-1}(w) = w^{d^n}$$

Equipotential Lines

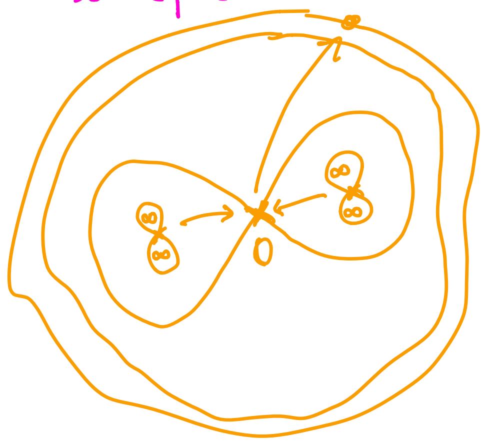
$\{z : G(z) = r\}$.

$\boxed{Q}$ Shape (Topology) of Equipotential Lines?

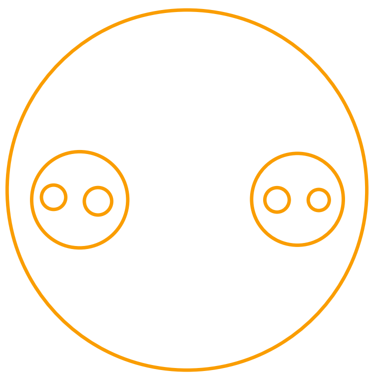
Rmk $\quad G(f(z)) = d\, G(z)$

$f_c(z) = z^2 + c$, $|c| >> 1$

Case1: critical point escapes $\quad f_c(z) = z^2 + 100$

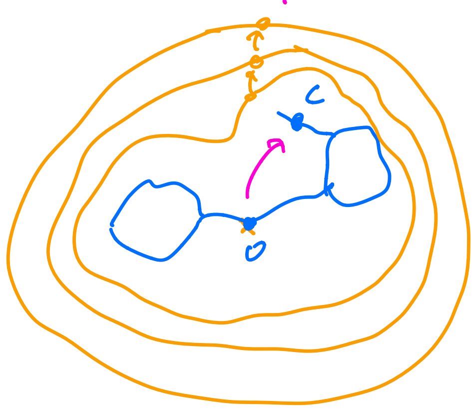


$0 \notin K(f)$



$J(f) = K(f)$ is a Cantor set

Case 2: critical point does <u>not</u> escape



If $0 \in K(f)$

Consider $V(r) := \{ z : G(z) \leq r \}$

Since $c \in V(r)$, $f^{-1}(V(r))$ is connected and

$$f : f^{-1}(V(r)) \longrightarrow V(r)$$

is branched

Hence: $K(f) = \bigcap_{r>0} V(r)$ is the intersection of a nested sequence of disks, hence it is connected

Def.: A compact set is cellular if it is the intersection of a nested sequence of topological disks.

Theorem If f is a quadratic polynomial, then either:

1. $0 \in K(f)$, hence $K(f)$ and $J(f)$ are connected
2. $0 \notin K(f)$, and $K(f) = J(f)$ is a Cantor set.

Cor.: $c \in M$ if and only if $K(f_c)$ is connected

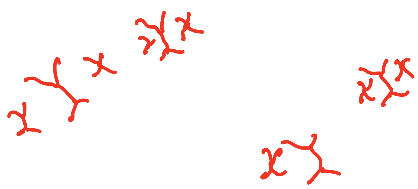
M is the "connectedness locus"

Note: $K(f) \ni 0 \longleftrightarrow K(f)$ connected.

Theorem If f is a polynomial of degree $d \geq 2$, then either:

① $K(f)$ contains all critical points and is cellular (hence connected)

② at least one critical point escapes, then $K(f)$ has uncountably many components (not necessarily true that $K(f)$ is Cantor set)



Theorem If f is hyperbolic (or subhyperbolic) then $K(f)$ is locally connected.

Cor.: We can understand $K(f)$ by quotienting the closed disk by a lamination.

Theorem If $K(f)$ is connected, then external rays for rational angles land at periodic points which are repelling or parabolic, or their preimages.

Theorem If the Julia set $J(f)$ is locally connected, then every periodic point <u>in</u> $J(f)$ is either repelling or parabolic.

Cor.: A Julia set with a Cremer point is <u>not</u> locally connected.

E.g.:

$$f(z) = e^{2\pi i\theta} z + z^2$$

θ is Liouville (e.g. $\theta = \sum_{n=1}^{\infty} 10^{-n!}$)

very close to being rational